Hale School

Mathematics Specialist

Test 5 --- Term 3 2018

Applications of Differentiation and Modelling Motion

Name: Moe Shun

/ 42

Instructions:

- **Calculators are NOT allowed**
- **External notes are not allowed**
- **Duration of test: 45 minutes**
- **Show your working clearly**
- **Use the method specified (if any) in the question to show your working (Otherwise, no marks awarded)**
- **This test contributes to 7% of the year (school) mark**

Use exact values in your answers.

Question 1 **(3, 3 = 6 marks)**

Differentiate the following equations with respect to x .

Please note that you <u>do not</u> need to simplify nor write explicitly in terms of $\frac{dy}{dx}$.

(a) $xy + x^3 = (1+y)^2$

$$x \cdot \frac{dy}{dx} + 1 \cdot y + 3x^2 = 2(1+y)^1 \frac{dy}{dx}$$

(b) $\frac{1}{\tan y} + x^2 y = \pi$

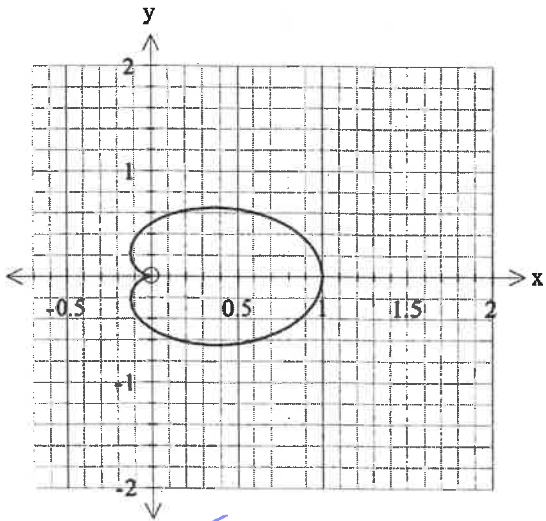
$$-\frac{1}{\tan^2 y} \cdot \frac{1}{\cos^2 y} \cdot \frac{dy}{dx} + 2xy + \frac{dy}{dx} x^2 = 0$$

Question 2 **(5 marks)**

Find the equation of the tangent line to the cardioid curve

$$x^2 + y^2 = (2x^2 + 2y^2 - x)^2 \text{ at the point } \left(0, \frac{1}{2}\right).$$

(A cardiod is a heart shaped curve shown below)



$$\underbrace{2x + 2y\frac{dy}{dx}}_{\checkmark} = \underbrace{2(2x^2 + 2y^2 - x)\left(4x + 4y\frac{dy}{dx} - 1\right)}_{\checkmark}$$

$$2(0) + 2\left(\tfrac{1}{2}\right)\frac{dy}{dx} = 2\left(2(0)^2 + 2\left(\tfrac{1}{2}\right)^2 - 0\right)\left(4(0) + 4\left(\tfrac{1}{2}\right)\frac{dy}{dx} - 1\right)$$

$$\frac{dy}{dx} = 2\left(\tfrac{1}{2}\right)\left(2\frac{dy}{dx} - 1\right)$$ ✓ substitutes $(0, \frac{1}{2}$

$$\frac{dy}{dx} = 2\frac{dy}{dx} - 1$$

$$\frac{dy}{dx} = 1$$ ✓ determines $\frac{dy}{dx}$

$$y = x + \frac{1}{2}$$ ✓ Eq^{n} of line

Question 3 **(5 marks)**

A mass has acceleration $a \ m/s^2$ given by $a = v^2 - 3$, where $v \ m/s$ is the velocity of the mass when it has a displacement of $x \ m$ from the origin,

Find v in terms of x given that $v = -2 \ m/s$ where $x = 1 \ m$.

$$a = v^2 - 3$$

$$\frac{dv}{dt} = v^2 - 3$$

$$v \frac{dv}{dx} = v^2 - 3$$ ✓ $v \frac{dv}{dx}$

$$\int \frac{v}{v^2-3} \, dv = \int 1 \, dx$$ ✓ separating variables

$$\frac{1}{2} \ln |v^2 - 3| = x + c$$ $v=-2$ $x=1$ ✓ integrating

$$\frac{1}{2} \ln |4-3| = 1 + c$$ ✓ solve for c

$$c = -1$$

$$\therefore \frac{1}{2} \ln |v^2 - 3| = x - 1$$

$$|v^2 - 3| = e^{2x-2}$$

$$v^2 - 3 = \pm e^{2x-2}$$

$$v^2 = 3 \pm e^{2x-2}$$

$$v = \pm \sqrt{3 \pm e^{2x-2}}$$ ✓ final expression.

as $v = -2$ when $x = 1$

$$v = -\sqrt{3 + e^{2x-2}}$$ ✓

Question 4 (3, 2, 2 = 7 marks)

A body moves such that its displacement from some fixed point O at time t seconds is given by $x = 3 + 6\cos \pi t$.

(a) Using the substitution $b = x - 3$, show that the motion is simple harmonic.

let $b = 6\cos \pi t$ ✓ sub correctly

$\dot{b} = -6\pi \sin \pi t$ ✓ $\dot{b}$

$\ddot{b} = -6\pi^2 \cos \pi t$

$\ddot{b} = -\pi^2 b$ ✓ $\ddot{b} = -\pi^2 b$

$\therefore$ Motion SHM

(b) What is the period and amplitude of the motion?

Amp = 6 units. ✓

Period $= \frac{2\pi}{\pi} = 2$ sec ✓

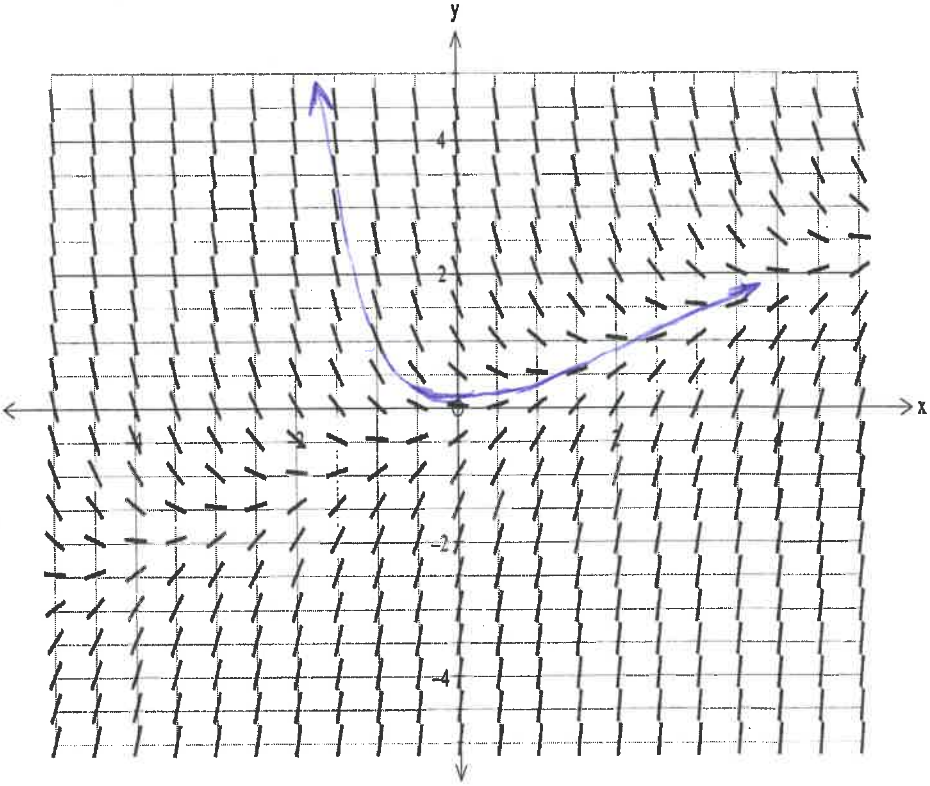
(c) Write an expression to determine the distance travelled by the body in the first 10 secs of motion. (You are not required to evaluate.)

$d = \int_a^b |v| \, dt$

$d = \int_0^{10} |-6\pi \sin \pi t| \, dt$ ✓ $\int |v|$ ✓ bounds

Question 5 (2, 2 = 4 marks)

A first-order differential equation has a slope field as shown in the diagram below.



✓ 1st Quad
✓ 2nd Quad
(accuracy)

(a) On the above slope field, draw in the curve representing the particular solution with initial condition (-1, 1).

(b) Determine with reasoning, which of the following equations best describes the differential equation.

A. $\frac{dy}{dx} = 2x + y$ B. $\frac{dy}{dx} = \frac{y}{2x}$ C. $\frac{dy}{dx} = \frac{2x}{y}$ D. $\frac{dy}{dx} = x - 2y$

× All gradient in 2nd Quad -ive

× $x=0$ changing grad

× $y=0$ changing gradient.

✓ answer

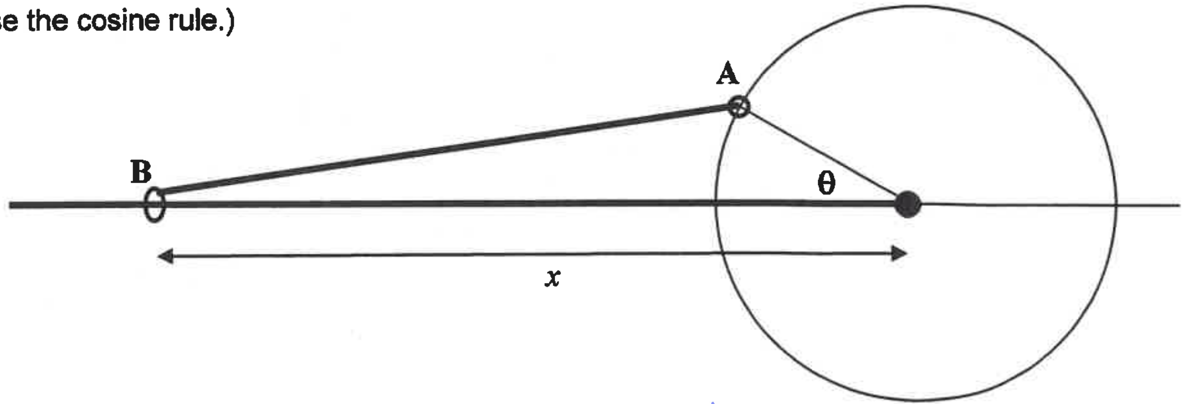
✓ reasoning.

Question 6 **(7 marks)**

A rigid rod AB, of length 25 cm, is attached at end A to a circular wheel of radius 7 cm that is turning at 0.5 radians per second. The other end B is attached to a ring that is free to slide along a second horizontal rod.

Determine how fast the ring is moving at the instant when A is in its highest position.

(HINT: Use the cosine rule.)



$\frac{d\theta}{dt} = 0.5$ rad/sec $\boxed{\checkmark \frac{d\theta}{dt}}$

highest position

$\theta = \frac{\pi}{2}$, $x = 24$ $\boxed{\checkmark}$

$25^2 = 7^2 + x^2 - 2 \times x \times 7 \times \cos\theta$ $\boxed{\checkmark \text{ cos rule.}}$

Diff wrt t: $0 = 0 + 2x\frac{dx}{dt} - 14\frac{dx}{dt}\cos\theta + 14x\sin\theta\frac{d\theta}{dt}$

$0 = 2 \times 24 \times \frac{dx}{dt} - 14\frac{dx}{dt}\cos\frac{\pi}{2} + 14(24)\sin\frac{\pi}{2}\,0.5$

$0 = 2 \times \cancel{24} \times \frac{dx}{dt} + 14(\cancel{24})\frac{1}{2}$

$\frac{dx}{dt} = -\frac{7}{2}$ cm/s

✓ diff wrt t
✓ correct diff.

✓ sub values

i.e. 3.5 cm/s towards wheel. $\boxed{\checkmark \text{ sol}^{n}}$

Question 7 **(4, 1, 3 = 8 marks)**

When a subject ends, students start to forget the material they have learned. The Ebbinghaus Forgetting Curve assumes that the rate at which a student forgets material is proportional to the difference between the material they currently remember and a positive constant *a*.
Thus, if $y = f(t)$ is the fraction of the original material remembered *t* weeks after a subject has ended, then *y* satisfies the equation:

$$\frac{dy}{dt} = -k(y-a)$$

where k is a positive constant, and a represents the fraction of the original material that will never be forgotten.

(a) Use separation of variables to establish $y = a + Ae^{-kt}$.

$$\int \frac{1}{y-a}\,dy = \int -k\,dt$$ ✓ separates variables

$$\ln|y-a| = -kt + c$$ ✓ integrates

$$y - a = \pm e^{-kt+c}$$ ✓

$$y = a + Ae^{-kt}$$ where $A = \pm e^{c}$

✓ defines A.

(b) Given $y(0) = 1$, find A in terms of a.

$$1 = a + A$$

$$A = 1 - a$$ ✓

(c) Suppose that one week after the final Specialist exam, a student can remember 75% of the material they knew when they sat the exam, and that 25% of the material will never be forgotten. What fraction of the material will they be able to remember 2 weeks after the Specialist exam?

$$y = \frac{1}{4} + \frac{3}{4}e^{-kt}$$ ✓ determines A & a

$\left(y = \frac{3}{4}\right)$ ✓

$$\frac{3}{4} = \frac{1}{4} + \frac{3}{4}e^{-k}$$

$$\frac{2}{4} = \frac{3}{4}e^{-k}$$

$$\frac{2}{3} = e^{-k}$$

At $t = 2$

$$y = \frac{1}{4} + \frac{3}{4}\left(e^{-k}\right)^2$$

$$y = \frac{1}{4} + \frac{3}{4} \times \left(\frac{2}{3}\right)^2$$

$$y = \frac{7}{12}$$ ✓ sol^{n}